Journal of Statistical Physics, Vol. 37, Nos. 1/2, 1984

Random-Walk Model for Equilibrium Resistance Fluctuations

C. J. Stanton[1] and Mark Nelkin[2]

Received January 20, 1984

The Voss–Clarke relation between the fluctuations of the band-limited Johnson noise power and the nonequilibrium steady-state current fluctuations is investigated for a continuous time random walk (CTRW). The equivalence between the CTRW and a multistate trapping model (MSTM), as shown by Kehr and Haus, is exploited to calculate the higher-order current correlation functions using the Markovian property of the MSTM description. The Voss–Clarke relation is found to be obeyed provided the trapping and release times are long compared to the microscopic correlation time of a carrier in the conduction state. The equilibrium resistance fluctuations are shown to arise physically from equilibrium fluctuations in the number of carriers in the conduction state. It is suggested that the results obtained here should hold for more general trapping models under the same physical conditions.

KEY WORDS: Random walk; equilibrium resistance fluctuations, $1/f$ noise; Voss–Clarke relation.

1. INTRODUCTION

When a constant current is passed through a resistor R, the voltage across the resistor fluctuates about its mean value $\bar{V}$. In many types of conductors,[1,2] the spectrum of these voltage fluctuations is observed to have the form

$$S_V(\omega) = 2\int_{-\infty}^{\infty} \langle [V(t) - \bar{V}][V(0) - \bar{V}]\rangle e^{i\omega t}\, dt = 4kTR + \bar{V}^2 S_1(\omega) \quad (1.1)$$

[1] Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853. Supported by a Fannie and John Hertz Foundation Fellowship and by the Semiconductor Research Corporation.

[2] School of Applied and Engineering Physics, Cornell University, Ithaca, New York 14853. Supported by the National Science Foundation through grant No. DMR-81-08328 and through the Cornell Materials Science Center.

822/37/1-2-1

Similarly, if a constant voltage is applied across the resistor, the spectrum of current fluctuations is found to be

$$S_I(\omega) = 4kT/R + \bar{I}^2 S_1(\omega) \tag{1.2}$$

where $\bar{I}$ is the mean current. The first term in (1.1) and (1.2) is the equilibrium Johnson noise. This term is well understood and is present even if there is no mean voltage across the resistor. The second term, $S_1(\omega)$, is referred to as the excess noise since it is not present if the mean voltage across the resistor is zero. For a wide variety of materials,[1,2] the excess noise $S_1(\omega)$ is found to be independent of $\bar{I}$ and $\bar{V}$ and varies approximately as $1/f$ (where $f = \omega/2\pi$ is the frequency) over several decades of frequency. In this paper, we are not concerned with trying to understand the physical origin of this frequency dependence, but instead want to clarify in what sense these excess fluctuations correspond to an equilibrium phenomenon, i.e., in what sense are they observable with no applied voltage.

The independence of $S_1(\omega)$ on the mean current or voltage suggests through Ohm's law that the fluctuations originate in the resistance:

$$S_1(\omega) = \frac{S_R(\omega)}{R^2} \tag{1.3}$$

In this case, the current does not generate the fluctuations but is only needed to probe them. In principle, these resistance fluctuations should be present in equilibrium with no net current, though not observable in the correlation functions (1.1) and (1.2). How then can these fluctuations be observed in equilbrium? Since the band-limited Johnson noise power in a bandwidth Δf has an average value of $4kTR(\Delta f)$, Voss and Clarke[3] suggested that these resistance fluctuations should manifest themselves as slow fluctuations in the average band-limited Johnson noise power. They showed this to be experimentally true for two $1/f$ noise sources[3] (discontinuous Nb films and InSb). Later Beck and Spruit[4] verified this result for a carbon resistor, another $1/f$ noise source.

The concept of equilibrium resistance fluctuations, however, has no precise statistical mechanical foundation. In an earlier paper,[5] Tremblay and Nelkin explored this question in the context of a nonlinear Langevin model. They found that a plausible but not unique model agreed with the Voss and Clarke idea.

In this paper, we examine the Voss–Clarke idea in the context of random walk models of current fluctuations. In Section 2 we show how the fluctuations in the band-limited Johnson noise can be expressed naturally in terms of the displacement $\alpha(t)$ of the random walker. The quantity of primary interest is

$$\langle [\alpha(t+\tau') - \alpha(t)]^2 [\alpha(\tau) - \alpha(0)]^2 \rangle \equiv 4\tau\tau' \langle D(0)\, D(t) \rangle \tag{1.4}$$

and can be thought of as defining equilibrium fluctuations of the diffusion coefficient. In (1.4), τ and τ' must be large enough to be in a diffusive regime for the mean square displacement, but both must be small compared to t.

In Section 3 we consider a continuous time random walk (CTRW) first applied to $1/f$ noise by Tunaley,[6] and recently extended by Nelkin and Harrison.[7] We exploit the equivalence between the CTRW and an M-state trapping model in the form given by Kehr and Haus.[8] This latter description gives an underlying Markovian description and allows the correlation function (1.4) to be calculated. We find agreement with the Voss–Clarke idea. Finally, in Section 4 we give a direct physical interpretation of the excess noise in terms of equilibrium fluctuations in the number of carriers in the conduction state.

2. FLUCTUATIONS IN THE JOHNSON NOISE POWER

In a random walk model, it is more convenient to discuss current fluctuations rather than voltage fluctuations. Although the original experiment was done with voltage fluctuations, there is no theoretical reason why it could not be done with current fluctuations. We thus extend the analysis of Tremblay and Nelkin,[5] essentially repeating their results (2.1)–(2.6), (3.22) with voltage fluctuations replaced by current fluctuations. We then go on to relate the current fluctuations to the displacement of the particle in the random walk.

In the experiment, the current is passed through a bandpass filter peaked at ω_0 with width $2(2\pi\Delta f)$. The filtered current is given by

$$I_{\omega_0}(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} G(\omega) I_\omega \tag{2.1}$$

where $G(\omega)$ is the filter function with properties

$$\begin{gathered} G(-\omega) = G^*(\omega) \\ (2\pi)^{-1} \int_{-\infty}^{\infty} d\omega\, |G(\omega)|^2 = 2\Delta f \end{gathered} \tag{2.2}$$

The "instantaneous Johnson noise power" is the filtered current squared and averaged over an interval $\Delta t \gtrsim 1/\omega_0$:

$$P_J(t) = \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} I^2_{\omega_0}(t')\, dt' \tag{2.3}$$

It is this quantity that Voss and Clarke originally proposed should exhibit the same fluctuations as the nonequilibrium excess current fluctuations. Its power spectrum is given by

$$S_P(\omega) \equiv 2\mathscr{F}\{\langle P_J(t)\, P_J(0)\rangle\}$$
$$\equiv 2\mathscr{F}\left\{\iiiint dt_1\, dt_2\, dt_3\, dt_4\; g(-t_1)\, g(-t_2)\, gt - t_3)\, g(t - t_4) \times \langle I(t_1)\, I(t_2)\, I(t_3)\, I(t_4)\rangle\right\} \qquad (2.4)$$

where $\mathscr{F}$ stands for Fourier transformation and $g(t)$ is the inverse-Fourier transform of $G(\omega)$.

When $S_p(\omega)$ was normalized by the average band-limited Johnson noise power

$$\langle P_J\rangle = S_I(\omega_0)\,\Delta f = \frac{4kT}{R}\Delta f \qquad (2.5)$$

Voss and Clarke observed a white background at high frequencies:

$$\frac{S_p(\omega)}{\langle P_J\rangle^2} \sim \frac{1}{\Delta f} \qquad (2.6)$$

In addition, at low frequencies, they observed an extra noise which they interpreted as resistance fluctuations and which satisfied

$$\frac{S_p(\omega)}{\langle P_J\rangle^2} - \frac{1}{\Delta f} = S_1(\omega) \qquad (2.7)$$

where $S_1(\omega)$ is defined by (1.1) and (1.2). Henceforth when we refer to $S_P(\omega)$, we shall mean only the excess noise above the white background noise (2.6).

In a random walk, to relate the spatially averaged current $I(t)$ to the motion of the charge carriers, we write

$$I(t) = \frac{e}{l} \sum_{i=1}^{N} v_i(t) \qquad (2.8)$$

where e is the electronic charge, l is the length of the sample, and $v_i(t)$ is the x component of the velocity of the ith charge carrier at time t. We treat the N charge carriers as independent and identically distributed to obtain

$$\langle I(t_1)\,I(t_2)\,I(t_3)\,I(t_4)\rangle = \frac{Ne^4}{l^4}\langle v(t_1)\,v(t_2)\,v(t_3)\,v(t_4)\rangle$$

$$+\frac{N(N-1)e^4}{l^4}\{\langle v(t_1)\,v(t_3)\rangle\langle v(t_2)\,v(t_4)\rangle + \langle v(t_1)\,v(t_2)\rangle\langle v(t_3)\,v(t_4)\rangle$$

$$+\langle v(t_1)\,v(t_4)\rangle\langle v(t_2)\,v(t_3)\rangle\} \tag{2.9}$$

The terms proportional to $N(N-1)$ in (2.9) are terms that would result in a Gaussian decomposition of the current. As discussed by Tremblay and Nelkin,[5] they only contribute to the white noise background in (2.6). Only the first term is of interest here. To evaluate it approximately, we let $t_2 = t_1' + t_1$, $t_4 = t_3 + t_3'$ so that (2.4) becomes

$$S_P(\omega) = \frac{2Ne^4}{l^4}\mathscr{F}\left\{4\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_3 \int_0^{\infty} dt_1' \int_0^{\infty} dt_3'\right.$$

$$\times\,[\langle v(t_1)\,v(t_1+t_1')\,v(t_3)\,v(t_3+t_3')\rangle$$

$$\left.\times\, g(-t_1)\,g(-t_1-t_1')\,g(r-t_3)\,g(t-t_3-t_3')]\right\} \tag{2.10}$$

The main contribution to (2.10) is when t_1' and t_3' are less than T, where T is a microscopic correlation time, i.e., the time during which the velocity of a particle remains correlated on a microscopic scale. The meaning of T will become clearer later. If we choose Δf such that

$$T \ll \frac{1}{\Delta f} \ll t \tag{2.11}$$

and provided the filter functions $g(t)$ are slowly varying over a time scale T, then (2.10) reduces to

$$S_P(\omega) = \frac{2Ne^4}{l^4}\mathscr{F}\left\{4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1\,dt_3\,g^2(t_1)\,g^2(t-t_3)\langle D(t_1)\,D(t_3)\rangle\right\} \tag{2.12}$$

Here

$$\langle D(t_1)\,D(t_3)\rangle = \int_0^{\infty} dt_1' \int_0^{\infty} dt_3' \langle v(t_1)\,v(t_1+t_1')\,v(t_3)\,v(t_3+t_3')\rangle \tag{2.13}$$

By (2.11), we see that $\langle D(t_1)\,D(t_3)\rangle$ is slowly varying compared to $g^2(t_1)\,g^2(t-t_3)$. Thus from (2.2) and Parseval's theorem,

$$S_P(\omega) = \frac{8Ne^4}{l^4}(2\Delta f)^2\,\mathscr{F}\{\langle D(0)\,D(t)\rangle\} \tag{2.14}$$

The fact that (2.13) agrees with (1.4) can be seen if one differentiates (1.4) with respect to τ, τ', expresses

$$\alpha(\tau) - \alpha(0) = \int_0^\tau dt\, v(t) \tag{2.15}$$

and uses the fact that since τ is of order T, one can extend the integral to ∞ with negligible additional contribution. Using (1.4) in (2.14), we finally obtain

$$S_p(\omega) = \frac{8Ne^4}{l^4} \frac{(\Delta f)^2}{\tau\tau'} \mathscr{F}(\langle [\alpha(\tau) - \alpha(0)]^2 [\alpha(t+\tau') - \alpha(t)]^2 \rangle) \tag{2.16}$$

It is this quantity which we wish to calculate for the random walk model in the next section. Even though jumps in a random walk are "instantaneous" so that the velocity $v(t)$ is not well defined, the correlation function (2.4) is well defined in terms of (2.16) and the particle displacements.

3. RANDOM-WALK MODEL

Tunaley[6] calculated the current noise for a hopping conduction model based on the Montroll–Weiss[9] continuous time random walk (CTRW). The CTRW is a generalization of the ordinary random walk in which particles no longer wait a fixed time interval before hopping to a new site, but wait a random time t given by a "waiting time distribution" $\psi(t)$. The average waiting time $\bar{t}$ is given by

$$\bar{t} = \int_0^\infty t\psi(t)\, dt \tag{3.1}$$

For the special case $\psi(t) = \delta(t - \bar{t})$ the CTRW reduces to the ordinary random walk.

Tunaley[6] found that in the CTRW, (1.2) was automatically satisfied and that the excess noise $S_1(\omega)$ was given by

$$S_1(\omega) = 2\bar{t}N^{-1}(1 + 2\,\mathrm{Re}\{[\tilde{\psi}^{-1}(i\omega) - 1]^{-1}\}) \tag{3.2}$$

where $\tilde{\psi}(i\omega)$ is the Laplace transform of the waiting time distribution

$$\tilde{\psi}(i\omega) = \int_0^\infty e^{-i\omega t} \psi(t)\, dt \tag{3.3}$$

The first term inside the brackets in (3.2) is an excess white noise term which has already been discussed by Tunaley[6] and Nelkin and Harrison.[7] This term contributes to the white background noise and we are interested in the excess noise above the background. Thus, if the Voss–Clarke idea applies to this model, then we should recover the same functional of the waiting time distribution which appears in the second term of (3.2).

In principle, the equilibrium CTRW is a well-defined stochastic process, and we should be able to calculate higher-order correlation functions such as (1.4). Our task is made much easier, however, by the knowledge that the CTRW is a contraction of an expanded description which is Markovian. Kehr and Haus[8] introduced a multistate trapping model (MSTM) described as follows (see Fig. 1). At each site of a lattice, there are M different states. The first state, $i = 1$, is the conduction state. Only in this state can particles hop between different sites. Hopping between sites occurs at a rate γ. The next $M - 1$ states ($i = 2,\dots, M$) are trap states. A particle is captured from the conduction state into the ith trap state at the same site with trapping rate γ_i and released from the ith trap into the conduction band with release rate r_i. There is no hopping between different trap states. Trapping and releasing occur only between a trap state and the conduction state at the same site. The trapping and release rates are the same at all sites.

The model is described by the Markovian master equation,

$$\frac{dP_1(\alpha, t)}{dt} = \gamma \sum_{\alpha'} [p(\alpha - \alpha') - \delta_{\alpha,\alpha'}] P_1(\alpha', t) - \sum_{i=2}^{M} \gamma_i P_1(\alpha, t) + \sum_{i=2}^{M} r_i P_i(\alpha, t) \tag{3.4}$$

$$\frac{dP_i(\alpha, t)}{dt} = -r_i P_i(\alpha, t) + \gamma_i P_1(\alpha, t), \qquad i = 2,\dots, M$$

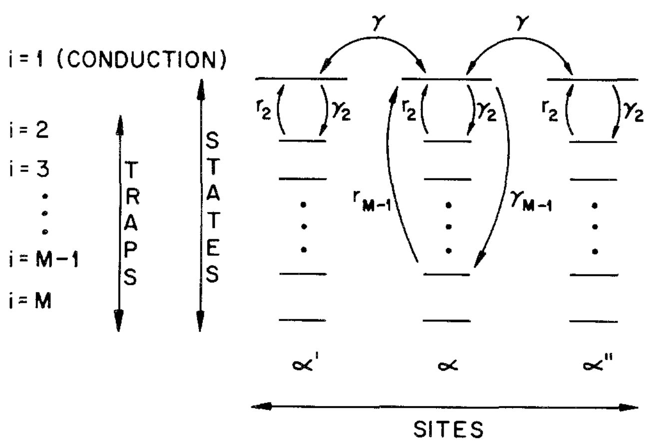

Fig. 1. The multistate trapping model (MSTM). Each site α contains M states. The first state, $i = 1$, is the conduction state. In this state, particles hop between sites at a rate γ. The states $i = 2,\dots, M$ are trap states. A particle is trapped and released between the ith trap state and the conduction state at the same site with rates γ_i, r_i. There is no hopping between traps.

where $P_i(\alpha, t)$ is the probability of being at site α at time t in state i for some given initial condition. γ_i and r_i are the capture and release rates for the ith trap state. $p(\alpha - \alpha')$ is the probability that if a particle at site α makes a hop, it hops into site α'.

One can introduce the probability that a particle is at a given site at time t without specifying the exact state it is in. This is just

$$P(\alpha, t) = \sum_{i=1}^{M} P_i(\alpha, t) \tag{3.5}$$

This description is referred to as the contracted description. Kehr and Haus[8] showed that the CTRW was equivalent to the contracted description of the MSTM through the relations

$$\begin{aligned} \tilde{\psi}(i\omega) &= \gamma[i\omega F(i\omega) + \gamma]^{-1} \\ F(z) &= 1 + \sum_{i=2}^{M} \frac{\gamma_i}{z + r_i} \end{aligned} \tag{3.6}$$

For this equivalence to hold,[8] it is essential that the MSTM be calculated for "equilibrium initial conditions"

$$P_i(\alpha, t=0) = \rho_i P(\alpha, t=0) \qquad \text{(for all } \alpha) \tag{3.7}$$

with ρ_i given by

$$\begin{aligned} \rho_1 &= [F(0)]^{-1} \\ \rho_i &= \frac{\gamma_i}{r_i}[F(0)]^{-1} \qquad i = 2, \ldots, M \end{aligned} \tag{3.8}$$

ρ_i gives the relative probability of finding a particle in state i at long times[8] and hence the name "equilibrium initial conditions."

As shown in Section 2, calculating the fluctuations in the band-limited Johnson noise power involves correlation functions of the form (1.4). To calculate them requires a knowledge of the joint distribution function

$$P(\alpha_4 t_4, \alpha_3 t_3, \alpha_2 t_2, \alpha_1 t_1) = \{\text{probability of being at site } \alpha_4 \text{ at } t_4 \textit{ and } \alpha_3 \text{ at } t_3 \textit{ and} \ldots\} \tag{3.9}$$

Since the expanded description of the MSTM is Markovian, this can easily be expressed as

$$P(\alpha_4 t_4, \alpha_3 t_3, \alpha_2 t_2, \alpha_1 t_1) = \sum_{ijkl} P(\alpha_4^i t_4 \mid a_3^j t_3)\, P(\alpha_3^j t_3 \mid \alpha_2^k t_2)\, P(\alpha_2^k t_2 \mid \alpha_1^l t_1)\, P_l(\alpha, t_1) \tag{3.10}$$

where $P(\alpha_4^i t_4 \mid \alpha_3^j t_3)$ is the conditional probability of a particle being at site α_4 in state i at t_4 given that it was at site α_3 in state j at t_3. $P_l(\alpha_1 t_1)$ is the same as in (3.4). For stationary, lattice translationally invariant systems, $P(\alpha_2^i t_2 \mid \alpha_1^j t_1)$ depends only on differences $\alpha = \alpha_2 - \alpha_1$ and $t = t_2 - t_1$. We denote this

$$P^{ij}(\alpha, t) \equiv P(\alpha_2^i t_2 \mid \alpha_1^j t_1) \tag{3.11}$$

The transition probabilities can be related to solutions of Eq. (3.4). Fourier–Laplace transforming (3.4) we obtain

$$[z\mathsf{I} + \mathsf{D}(k)]\, \tilde{\mathbf{P}}(k, z) = \mathbf{P}(k, 0) \tag{3.12}$$

where $\tilde{\mathbf{P}}(k, z)$ is an M-dimensional vector whose entries are the various states

$$\tilde{\mathbf{P}}(k, z) = \begin{pmatrix} \tilde{P}_1(k, z) \\ \tilde{P}_2(k, z) \\ \vdots \\ \tilde{P}_M(k, z) \end{pmatrix}$$

Here the tilde denotes Laplace transform, with z the Laplace transform variable. $\mathbf{P}(k, 0)$ is the N-dimensional vector of initial conditions. I is the $M \times M$ unit matrix and $\mathsf{D}(k)$ is the dynamical matrix given by

$$\begin{aligned} [\mathsf{D}(k)]_{11} &= \gamma[1 - \lambda(k)] + \sum_{i=2}^{M} \gamma_i \\ [\mathsf{D}(k)]_{1i} &= -r_i, \qquad i = 2,\ldots, M \\ [\mathsf{D}(k)]_{i1} &= -\gamma_i, \qquad i = 2,\ldots, M \\ [\mathsf{D}(k)]_{ij} &= r_i \delta_{ij}, \qquad i, j = 2,\ldots, M \end{aligned} \tag{3.13}$$

$\lambda(k)$ is the Fourier transform of $p(\alpha - \alpha')$.

The formal solution to (3.12) is

$$\tilde{\mathbf{P}}(k, z) = \mathsf{G}(k, z)\, \mathbf{P}(k, 0) \tag{3.14}$$

with

$$\mathsf{G}(k, z) = [z\mathsf{I} + \mathsf{D}(k)]^{-1} \tag{3.15}$$

Note that (3.14) shows the explicit dependence of $\tilde{\mathbf{P}}(k, z)$ on the initial conditions $\mathbf{P}(k, 0)$. It is easy to see from (3.14) that the Fourier–Laplace transform of the transition probabilities are given by

$$\mathscr{F} - \mathscr{L}(P^{ij}(\alpha, t)) = [\mathsf{G}(k, z)]_{ij} \tag{3.16}$$

where $\mathscr{L}$ stands for Laplace transform. To find the transition probabilities we must thus solve (3.15). This is outlined in the appendix, where we find

$$\mathsf{G}(k, z) = \{zF(z) - \gamma[\lambda(k) - 1]\}^{-1}\,\mathsf{A}(k, z) \tag{3.17}$$

with

$$\begin{aligned}
&[\mathsf{A}(k, z)]_{11} = 1\\
&[\mathsf{A}(k, z)]_{1i} = \frac{r_i}{z + r_i}, \qquad i = 2,\dots, M\\
&[\mathsf{A}(k, z)]_{i1} = \frac{\gamma_i}{r_i}, \qquad i = 2,\dots, M\\
&[A(k, z)]_{ij} = [(z + r_i)(z + r_j)]^{-1}\{\gamma_i r_j + \delta_{ij}[zF(z) - \gamma(\lambda(k) - 1)(z + r_i)]\}\\
&\hphantom{[A(k, z)]_{ij} = [(z + r_i)(z + r_j)]^{-1}\{\gamma_i r_j + \delta_{ij}[zF(z) - \gamma(\lambda(k) - 1)}\quad i, j = 2,\dots, M
\end{aligned} \tag{3.18}$$

Rather than inverse Fourier transforming (3.17) and (3.18) we find the correlation function (1.4) directly from the characteristic function:

$$\begin{aligned}
&\langle [\alpha(t_4) - \alpha(t_3)]^2 [\alpha(t_2) - \alpha(t_1)]^2 \rangle\\
&\quad = \left(\frac{\partial}{\partial_{q_4}} - \frac{\partial}{\partial_{q_3}}\right)^2 \left(\frac{\partial}{\partial_{q_2}} - \frac{\partial}{\partial_{q_1}}\right)^2 Z(q_1, q_2, q_3, q_4)\big|_{q_1 = q_2 = q_3 = q_4 = 0}
\end{aligned} \tag{3.19}$$

with

$$\begin{aligned}
Z = \iiint d\alpha_1\, d\alpha_2\, d\alpha_3\, d\alpha_4\, P(\alpha_4 t_4, \alpha_3 t_3, \alpha_2 t_2, \alpha_1 t_1)\\
\times \exp[i(q_1\alpha_1 + q_2\alpha_2 + q_3\alpha_3 + q_4\alpha_4)]
\end{aligned} \tag{3.20}$$

Without loss of generality,, we take $\alpha_1 = 0$ at $t_1 = 0$ so that

$$P_l(\alpha_1 t_1) = \rho_l \delta(\alpha_1 - 0) \tag{3.21}$$

Then using (3.10), (3.16), and making a change of variables, we find that (3.19) can be written in a compact matrix notation

$$\begin{aligned}
&\langle [\alpha(t + \tau') - \alpha(t)]^2 [\alpha(\tau) - \alpha(0)]^2 \rangle\\
&\quad = \frac{\partial^2}{\partial_{q^2}} \mathsf{f}(q, \tau)\big|_{q=0}\, \mathsf{f}(q, t)\big|_{q=0} \frac{\partial^2}{\partial_{q^2}} \mathsf{f}(q, \tau')\big|_{q=0}\, \boldsymbol{\rho}
\end{aligned} \tag{3.22}$$

with the matrix $\mathsf{f}(q, \tau)$ defined by

$$\mathsf{f}(q, \tau) = \mathscr{L}^{-1}(\mathsf{G}(q, z)) \tag{3.23}$$

and $\boldsymbol{\rho}$ the vector of initial probabilities

$$\boldsymbol{\rho} = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_M \end{pmatrix} \tag{3.24}$$

A direct calculation shows

$$\frac{\partial^2}{\partial_{q^2}} \mathsf{f}(q, t)|_{q=0} = \mathscr{L}^{-1}\left(\frac{-\gamma\sigma_0^2}{[zF(z)]^2}\,\mathsf{B}(z)\right) \tag{3.25}$$

with

$$\begin{aligned} &[\mathsf{B}(z)]_{11} = 1 \\ &[\mathsf{B}(z)]_{1i} = \frac{r_i}{z + r_i}, \qquad && i = 2,\ldots, M \\ &[\mathsf{B}(z)]_{i1} = \frac{\gamma_i}{z + r_i}, \qquad && i = 2,\ldots, M \\ &[\mathsf{B}(z)]_{ij} = \frac{\gamma_i r_j}{(z + r_i)(z + r_j)}, \qquad && i, j = 2,\ldots, M \end{aligned} \tag{3.26}$$

In obtaining (3.26), we have made use of

$$\begin{aligned} &\lambda(0) = 1 \\ &\frac{\partial}{\partial_q}\lambda(q)|_{q=0} = i\mu = 0 \qquad && \text{(for no field)} \\ &\frac{\partial^2}{\partial_{q^2}}\lambda(q)|_{q=0} = -\sigma_0^2 \qquad && \text{(for no field)} \end{aligned} \tag{3.27}$$

where μ and σ^2 are the mean and mean square displacement for a single jump, i.e.,

$$\mu = \int d\alpha\; \alpha p(\alpha), \qquad \sigma_0^2 = \int d\alpha\; \alpha^2 p(\alpha) \tag{3.28}$$

For τ, τ' small compared to the capture and release times, i.e.,

$$\tau, \tau' \ll \frac{1}{r_i}, \frac{1}{\gamma_i}, \quad \text{or} \quad z \gg r_i, \gamma_i \tag{3.29}$$

only the 1–1 entry of (3.25) contributes to lowest order in $r_i\tau$, $\gamma_i\tau$, i.e.,

$$\left[\frac{\partial^2}{\partial_{q^2}}\mathbf{f}(q,z)|_{q=0}\right]_{11} = -\gamma\sigma_0^2\tau$$
$$\left[\frac{\partial^2}{\partial_{q^2}}\mathbf{f}(q,\tau)|_{q=0}\right]_{ij} = O((r_i\tau)^2, (\gamma_i\tau)^2), \qquad (i,j)\neq(1,1) \tag{3.30}$$

The assumptions made in (3.30) are physically quite reasonable and will be discussed later. With the simplification (3.30), the matrix multiplication in (3.22) is easily performed:

$$\langle[\alpha(t+\tau')-\alpha(t)]^2[\alpha(\tau)-\alpha(0)]^2\rangle = \frac{\gamma^2\sigma_0^4\tau\tau'}{1+\sum(\gamma_i/r_i)}\mathscr{L}^{-1}([zF(z)]^{-1}) \tag{3.31}$$

From the equivalence between the CTRW and MSTM (3.6), it follows that

$$\langle[\alpha(t+\tau')-\alpha(t)]^2[\alpha(\tau)-\alpha(0)]^2\rangle = \frac{\sigma_0^4\tau\tau'}{\bar{t}}\mathscr{L}^{-1}\left(\frac{\tilde{\psi}(z)}{1-\tilde{\psi}(z)}\right) \tag{3.32}$$

Thus from (2.16) we find that the power spectrum of band-limited Johnson noise fluctuations is given by[3]

$$S_P(\omega) = \frac{16Ne^4\sigma_0^4(\Delta f)^2}{l^4\bar{t}}\operatorname{Re}\left[\frac{\tilde{\psi}(i\omega)}{1-\tilde{\psi}(i\omega)}\right] \tag{3.33}$$

From (2.5) we see that the average power is

$$\langle P_J\rangle = \frac{4kT\Delta f}{R} = \frac{2Ne^2\sigma_0^2}{l^2\bar{t}}\Delta f \tag{3.34}$$

The right-hand side of (3.34) has been shown to be true by Tunaley.[6] Combining (3.33) and (3.34) we finally obtain our result:

$$\frac{S_P(\omega)}{\langle P_J\rangle^2} = \frac{4\bar{t}}{N}\operatorname{Re}\left[\frac{\tilde{\psi}(i\omega)}{1-\tilde{\psi}(i\omega)}\right] = S_1(\omega) \tag{3.35}$$

The Voss–Clarke idea thus holds for the CTRW. Note that the result (3.35) holds even if $S_1(\omega)$ does not have a $1/f$ frequency dependence as in the original Voss–Clarke experiment. The spectrum, $S_1(\omega)$, obtained for various choices of the trapping parameters γ_i, r_i or the waiting time distribution $\tilde{\psi}(i\omega)$, has been considered by Nelkin and Harrison[7] and Tunaley.[6] The result (3.35) holds independently of the choice of parameters, provided $\bar{t}$ is finite.

[3] We make use of the fact that if $H(t)$ is an even function of t then $\mathscr{F}(H(t)) = 2\operatorname{Re}\mathscr{L}(H(t))|_{z=i\omega}$. Note that the function in (3.32) is even in t since it is proportional to $\langle I(0)\,I(t)\rangle$, which is even in t.

4. NUMBER FLUCTUATIONS

An interesting question to ask is "What is the power spectrum of fluctuations of the number of carriers in the conduction ($i = 1$) state?" The power spectrum for number fluctuations is defined as

$$S_N(\omega) = 2 \int dt\, e^{i\omega t} \langle N_1(0)\, N_1(t) \rangle \tag{4.1}$$

where $N_1(0)$ is the number of carriers in the conduction state at time 0 and $N_1(t)$ at time t. In the independent particle approximation,

$$\langle N_1(0)\, N_1(t) \rangle = N \times \{\text{probability of a particle being in the hopping state at times } t \text{ and } 0\} \tag{4.2}$$

This probability is given by

$$\sum_{\alpha, \alpha'} P(\alpha^1 t, \alpha'^1 0) \equiv \sum_{\alpha} P^{11}(\alpha, t) \rho_1 = \mathscr{L}^{-1}([\mathrm{G}(k, z)]_{11}|_{k=0} \rho_1) \tag{4.3}$$

From (3.17) and (3.18) we see

$$\langle N_1(0)\, N_1(t) \rangle = N\rho_1 [zF(z)]^{-1} \tag{4.4}$$

The average number of carriers in the conduction band is

$$\bar{N}_1 = N\rho_1 \tag{4.5}$$

Thus from (3.6)

$$\frac{S_{N_1}(\omega)}{(\bar{N}_1)^2} = \frac{2F(0)}{N[zF(z)]} = \frac{4\bar{t}\, \mathrm{Re}\{\tilde{\psi}(i\omega)/[1 - \tilde{\psi}(\omega)]\}}{N} \tag{4.6}$$

that is

$$\frac{S_{N_1}(\omega)}{\bar{N}_1^2} = \frac{S_p(\omega)}{\langle P_J \rangle^2} = S_1(\omega) \tag{4.7}$$

This gives us a simple interpretation of the fluctuations in the CTRW in terms of the MSTM. The fluctuations are due to fluctuations in the number of carriers in the conduction band. The presence of a field affects only the hopping between the various sites and not the transitions between the states at a given site. Hence, the fluctuations are trully "equilibrium resistance fluctuations" and it is not surprising that the Voss–Clarke result holds. The requirement that τ, τ' be small compared to the trapping and release times now seems physically reasonable. If the trapping and release times are not large compared to τ, τ', then the fluctuations in the number of carriers will be washed out since they occur on a time scale smaller than what we are observing. A microscopic correlation time T is seen to be the correlation time of a particle in the conduction band.

Again, note that Eq. (3.33) is not dependent on any particular form for the spectrum, which indicates that it should be possible for the Voss–Clarke result to hold for sources that do not have a $1/f$ spectrum.

5. DISCUSSION

It is physically plausible that the Voss–Clarke result holds for this model. One way of seeing this in terms of the MSTM is as follows: Consider the four-time velocity correlation function as in (2.13)

$$\langle v(0)\, v(\tau)\, v(t)\, v(t+\tau') \rangle \tag{5.1}$$

where τ, τ' are of the order of a microscopic correlation time T, and t is the order of the trapping and release rates,

$$t \gg T \gtrsim \tau, \tau' \tag{5.2}$$

Since t is much greater than T, we expect that $v(0)\, v(\tau)$ and $v(t)\, v(t+\tau)$ would be independent of each other and thus (5.1) would factor as

$$\langle v(0)\, v(\tau)\, v(t)\, v(t+\tau') \rangle = \langle v(0)\, v(\tau) \rangle \langle v(t)\, v(t+\tau') \rangle \tag{5.3}$$

This indeed would be the case if the particle were always in the conduction band and there were no traps. However, because there are traps, there is additional correlation since there is a probability that a particle in the conduction band at time 0 may not be in the conduction band at t. That is, the fact that a particle can be trapped reduces the correlation function (5.3) by a factor[4]

$$\frac{\langle N_1(0)\, N_1(t) \rangle}{(\bar{N}_1)^2} \tag{5.4}$$

the probability of being in the conduction state at time 0 and time t. Thus, (5.3) is

$$\langle v(0)\, v(\tau)\, v(t)\, v(t+\tau') \rangle = \langle v(0)\, v(\tau) \rangle \langle v(t)\, v(t+\tau') \rangle \frac{\langle N_1(0)\, N_1(t) \rangle}{\bar{N}_1^2} \tag{5.5}$$

[4] Actually, (5.3) should be reduced by a factor proportional to $\langle N_1(0)\, N_1(\tau)\, N_1(t)\, N_1(t+\tau') \rangle$, i.e., the probability of being in the conduction state at times 0, τ, t, $t+\tau'$. However, since τ, $\tau' \ll t$, we assume that a particle band at 0 will be there a time τ later with probability one, and similarly for t and $t+\tau'$.

When we normalize (5.5) by the square of the band-limited Johnson noise power, we essentially divide out by $\langle v(0)\, v(\tau)\rangle^2$; thus the correlation function (5.1), when suitably normalized, is actually a measure of the number fluctuations.

Now, let us consider the correlation function

$$\langle v(0)\, v(t)\rangle \tag{5.6}$$

Again t is large compared with the microscopic correlation time T. We thus expect (5.6) to factor into

$$\langle v(0)\, v(t)\rangle = \langle v(0)\rangle\langle v(t)\rangle \tag{5.7}$$

Again, this would be true if there were no traps but the presence of traps reduces (5.6) by $\langle N_1(0)\, N_1(t)\rangle/\bar{N}_1^2$. Hence, (5.6) becomes

$$\langle v(0)\, v(t)\rangle = \langle v(0)\rangle\langle v(t)\rangle\langle N_1(0)\, N_1(t)\rangle/\bar{N}_1^2 \tag{5.8}$$

Now with no applied field, $\langle v(0)\rangle = 0$ and (5.8) vanishes. With an applied field, $\langle v(0)\rangle \neq 0$, and when we normalize out by the current squared (essentially divide out by $\langle v(0)\rangle^2$) we see that (5.5) gives us a measure of the number fluctuations. The field does not cause the fluctuations but is only needed so that $\langle v(0)\rangle$ is not zero. However, since $\langle v(0)\, v(\tau)\rangle$ is not zero in (5.4), the four-point correlation function (5.1) does not need an applied field to be nonzero. Thus, in our model, the two-point correlation function with a field on and the four-point correlation function without a field measure the same thing, namely, the fluctuations in the number of carriers in the conduction band.

In view of this discussion, it is not surprising that the CTRW, which is equivalent to a trapping model, exhibits the Voss–Clarke result. Furthermore, the argument presented here seems fairly general and would appear to apply to any trapping model where the capture and release times are long compared to the microscopic correlation time of a particle in the conduction state. The result is not dependent on the specific nature of the spectrum and should hold for a non-$(1/f)$ spectrum. It should also hold for the physically relevant generalization where trapping parameters vary randomly from site to site.

ACKNOWLEDGMENTS

One of us (M.N.) would like to thank Andre-Marie Tremblay for a very useful discussion at the beginning of this work. The other (C.J.S.) would like to thank Joe Mantese and John Scofield for helpful conversations. This work was supported in part by a Hertz Foundation Fellowship (C.J.S.) and in part by the Semiconductor Research Corporation and the National Science Foundation through the Materials Science Center at Cornell University.

APPENDIX: CALCULATING $\mathrm{G}(k, z) = [z\mathrm{I} + \mathrm{D}(k)]^{-1}$

In this appendix, we invert the matrix $z\mathrm{I} + \mathrm{D}(k)$ to solve Eq. (3.15). The method we use is the standard algorithm of augmenting the matrix with the identity matrix and performing elementary row operations until the original matrix is conerted into the identity matrix. The augmented identity matrix will then be converted into the inverse matrix:

$$[\mathrm{A} \mid \mathrm{I}] \xrightarrow[\substack{\text{elementary}\\ \text{row}\\ \text{operations}}]{} [\mathrm{I} \mid \mathrm{A}^{-1}]$$

We start with $z\mathrm{I} + \mathrm{D}(k)$ given by (3.12) and (3.13) and perform the following sets of elementary row operations in order:

1. Multiply the ith row by $1/z + r_i$ for $i = 2,..., M$.
2. Add r_i times the ith row to row 1 for $i = 2,..., M$.
3. Divide row 1 by $zF(z) - \gamma[\lambda(k) - 1]$.
4. Add $\gamma_i/z + r_i$ times row 1 to row i ($i = 2,..., M$).

This gives us the matrix $\mathrm{G}(k, z)$ given by (3.17) and (3.18).

REFERENCES

1. P. Dutta and P. M. Horn, *Rev. Mod. Phys.* **53**:497 (1981).
2. F. N. Hooge, T. G. M. Kleinpenning, and L. K. J. Vandamme, *Rep. Prog. Phys.* **44**:479 (1981).
3. R. F. Voss and J. Clarke, *Phys. Rev. Lett.* **36**:42 (1976).
4. H. G. E. Beck and W. P. Spruit, *J. Appl. Phys.* **49**:3384.
5. A.-M. S. Tremblay and M. Nelkin, *Phys. Rev. B* **24**:2551 (1981).
6. J. K. E. Tunaley, *J. Stat. Phys.* **15**:149 (1976).
7. M. Nelkin and Alan K. Harrison, *Phys. Rev. B* **26**:6696 (1982).
8. K. W. Kehr and J. W. Haus, *Physica* **93A**:412 (1978). Also see the references within for more discussion on the relationship between master equations and random walks with internal states.
9. E. W. Montroll and G. M. Weiss, *J. Math. Phys.* **6**:167 (1965).